

Optimizing the Graph Minors Weak Structure Theorem

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Abstract

One of the major results of [N. Robertson and P. D. Seymour. *Graph minors. XIII. The disjoint paths problem. J. Combin. Theory Ser. B*, 63(1):65–110, 1995], also known as *the weak structure theorem*, revealed the local structure of graphs excluding some graph as a minor: each such graph G either has small treewidth or contains the subdivision of a wall that can be arranged “bidimensionally” inside G , given that some small set of vertices are removed. We prove an optimized version of that theorem where (i) the relation between the treewidth of the graph and the height of the wall is linear (thus best possible) and (ii) the number of vertices to be removed is minimized.

Keywords: Graph minors, Treewidth

1 Introduction

The Graph Minors series of Robertson and Seymour appeared to be a rich source of structural results in graph theory with multiple applications in algorithms. One of the most celebrated outcomes of this project was the existence of an $O(n^3)$ step algorithm for solving problems such as the DISJOINT PATH and the MINOR CONTAINMENT. A basic ingredient of this algorithm is a theorem, proved in paper XIII of the series [10], revealing the local structure of graphs excluding some graph as a minor. This result, now called the *weak structure theorem*, asserts that there is a function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that for every integer k , every h -vertex graph H , and every graph G , one of the following holds:

1. G contains H as a minor,
2. G has treewidth at most $f(k, h)$, or

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3. G contains a set X of at most $\binom{h}{2}$ vertices (called *apices*) such that $G \setminus X$ contains as a subgraph the subdivision W of a wall of height k that is arranged inside G in a “flat” manner (*flatness condition*).

To make the above statement precise we need to clarify the flatness condition in the third statement above. We postpone this complicated task until Section 3 and instead, we roughly visualize W in a way that the part of $G \setminus X$ that is located inside the perimeter P of W can be seen as a set of graphs attached on a plane region where each of these graphs has bounded treewidth and its boundary with the other graphs is bounded by 3.

The algorithmic applications of the weak structure theorem reside in the fact that the graph inside P can be seen as a *bidimensional structure* where, for several combinatorial problems, a solution certificate can be revised so that it avoids the middle vertex of the subdivided wall W . This is known as the *irrelevant vertex technique* and can be seen as a reduction of an instance of a problem to an equivalent one where this “irrelevant vertex” has been deleted. The application of this technique has now gone much further than its original use in the Graph Minors series and has evolved to a standard tool in algorithmic graph minors theory (see [1, 2, 5, 6, 7, 8] for applications of this technique).

In this paper we prove an optimized version of the weak structure theorem. Our improvement is twofold: first, the function f is now linear on k and second, the number of the apices is bounded by $h - 5$. Both our optimizations are optimal as indicated by the graph J obtained by taking a $(k \times k)$ -grid (for $k \geq 3$) and making all its vertices adjacent with a copy of K_{h-5} . Indeed, it is easy to verify that J excludes $H = K_h$ as a minor, its treewidth is $k + h - 5$ and becomes planar (here, this is equivalent to the flatness condition) after the removal of exactly $h - 5$ vertices.

Our proof deviates significantly from the one in [10]. It builds on the (strong) *structure theorem* of the Graph Minors that was proven in paper XVI of the series [11]. This theorem reveals the global structure of a graph without a K_h as a minor and asserts that each such graph can be obtained by gluing together graphs that can “almost be embedded” in a surface where K_h cannot be embedded (see 2 for the exact statement). The proof exploits this structural result and combines it with the fact, proved in [4], that apex-free “almost embedded graphs” without a $(k \times k)$ -grid have treewidth $O(k)$.

The organization of the paper is the following. In Section 2 we give the definitions of all the tools that we are going to use in our proof, including the Graph Minors structure theorem. The definition of the flatness condition is given in Section 3, together with the statement of our main result. Some lemmata concerning the invariance of the flatness property under certain local transformations are given in Section 4.1 and further definitions and results concerning apex vertices are given in Section 4.2. Finally, the proof of our main result is presented in Section 5.

2 Preliminaries

Let $n \in \mathbb{N}$. We denote by $[n]$ the set $\{1, 2, \dots, n\}$. Moreover, for every $k \leq n$, if S is a set such that $|S| = n$ we say that a set $S' \subseteq S$ is a k -subset of S if $|S'| = k$.

A *graph* G is a pair (V, E) where V is a finite set, called the *vertex set* and denoted by $V(G)$, and E is a set of 2-subsets of V , called the *edge set* and denoted by $E(G)$. We denote by $\mathbf{n}(G)$ the number of vertices of G , i.e. $\mathbf{n}(G) = |V(G)|$. If we allow E to be a subset of $\mathcal{P}(V)$ then we call the pair $H = (V(H), E(H))$ a *hypergraph*. The *incidence graph* of a hypergraph H is the bipartite graph $I(H)$ on the vertex set $V(H) \cup E(H)$ where $v \in V(H)$ is adjacent to $e \in E(H)$ if and only if $v \in e$, i.e. v is incident to e in H . We say that a hypergraph H is *planar* if its incidence graph is planar. Unless otherwise stated, we consider finite undirected graphs without loops or multiple edges.

Let G be a graph. For a vertex v , we denote by $N_G(v)$ its *(open) neighborhood*, i.e. the set of vertices which are adjacent to v . The *closed neighborhood* $N_G[v]$ of v is the set $N_G(v) \cup \{v\}$. For $U \subseteq V(G)$, we define $N_G[U] = \bigcup_{v \in U} N_G[v]$. We may omit the index if the graph under consideration is clear from the context. If $U \subseteq V(G)$ (resp. $u \in V(G)$ or $E \subseteq E(G)$ or $e \in E(G)$) then $G - U$ (resp. $G - u$ or $G - E$ or $G - e$) is the graph obtained from G by the removal of vertices of U (resp. of vertex u or edges of E or of the edge e). We say that a graph H is a subgraph of a graph G , denoted by $H \subseteq G$, if H can be obtained from G after deleting edges and vertices. Let \mathcal{C} be a class of graphs and S be a set of vertices. We denote by \mathbf{UC} the graph $\bigcup_{G \in \mathcal{C}} G$ and by $\mathcal{C} \setminus S = \{G \setminus S \mid G \in \mathcal{C}\}$.

The complete graph on n vertices is denoted by K_n . Moreover, if S is a finite set, we denote by $K[S]$ the complete graph with vertex set S . Let G be a graph such that $K_3 \subseteq G$ and x, y, z be the vertices of K_3 . The ΔY -transformation of K_3 in G is the following: We remove the edges $\{x, y\}, \{y, z\}, \{x, z\}$, add a new vertex w and then add the edges $\{x, w\}, \{y, w\}, \{z, w\}$.

Let G be a graph. We say that G is an apex graph if there exists a vertex $v \in V(G)$ such that $G \setminus v$ is planar. Moreover, we say that G is an α_G -apex graph if there exists an $S \subseteq V(G)$ such that $|S| \leq \alpha_G$ and $G \setminus S$ is planar. We denote by $\mathbf{an}(G)$, the minimum $k \in \mathbb{N}$ such that G is a k -apex graph, i.e. $\mathbf{an}(G) = \min\{k \in \mathbb{N} \mid \exists S \subseteq V(G) : (|S| \leq k \wedge G \setminus S \text{ is planar})\}$. Clearly, $\mathcal{G} = \{G \mid \mathbf{an}(G) = 1\}$ is the class of the apex graphs.

Observation 1. *Let T be a tree, $k \in \mathbb{N}$ and $w : V(T) \rightarrow \mathbb{N}$ such that there exists at least one vertex $v \in V(T)$ with $w(v) \geq k$. There exists a vertex $u \in V(T)$ such that at most one of the connected components of $(G \setminus u)$ contains a vertex u' with $w(u') > k$.*

Proof. Let $Y = \{v \in V(T) \mid w(v) \geq k\}$. Pick a vertex r of T and let v be a vertex of Y with maximum distance away from r . It is easy to verify that the lemma holds for v . \square

Surfaces. A *surface* Σ is a compact 2-manifold without boundary (we always consider connected surfaces). Whenever we refer to a Σ -embedded graph G we consider a 2-cell

embedding of G in Σ . To simplify notations, we do not distinguish between a vertex of G and the point of Σ used in the drawing to represent the vertex or between an edge and the line representing it. We also consider a graph G embedded in Σ as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H , where $H \subseteq G$. Recall that $\Delta \subseteq \Sigma$ is an open (resp. closed) disc if it is homeomorphic to $\{(x, y) : x^2 + y^2 < 1\}$ (resp. $\{(x, y) : x^2 + y^2 \leq 1\}$). The *Euler genus* of a non-orientable surface Σ is equal to the non-orientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The *Euler genus* of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ . We refer to the book of Mohar and Thomassen [9] for more details on graphs embeddings. The *Euler genus* of a graph G (denoted by $\mathbf{eg}(G)$) is the minimum integer γ such that G can be embedded on a surface of the Euler genus γ .

Contractions and minors. Given an edge $e = \{x, y\}$ of a graph G , the graph G/e is obtained from G by contracting the edge e , i.e. the endpoints x and y are replaced by a new vertex v_{xy} which is adjacent to the old neighbors of x and y (except x and y). A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G . An alternative, and more useful for our purposes, definition of a contraction is the following.

Let G and H be graphs and let $\phi : V(G) \rightarrow V(H)$ be a surjective mapping such that

1. for every vertex $v \in V(H)$, its codomain $\phi^{-1}(v)$ induces connected graph $G[\phi^{-1}(v)]$;
2. for every edge $\{v, u\} \in E(H)$, the graph $G[\phi^{-1}(v) \cup \phi^{-1}(u)]$ is connected;
3. for every $\{v, u\} \in E(G)$, either $\phi(v) = \phi(u)$, or $\{\phi(v), \phi(u)\} \in E(H)$.

We, then, say that H is a *contraction of G via ϕ* and denote it as $H \leq_c^\phi G$. Let us observe that H is a contraction of G if $H \leq_c^\phi G$ for some $\phi : V(G) \rightarrow V(H)$. In this case we simply write $H \leq_c G$. If $H \leq_c^\phi G$ and $v \in V(H)$, then we call the codomain $\phi^{-1}(v)$ the *model of v in G* .

Let G be a graph embedded in some surface Σ and let H be a contraction of G via function ϕ . We say that H is a *surface contraction* of G if for each vertex $v \in V(H)$, $G[\phi^{-1}(v)]$ is embedded in some open disk in Σ .

Let G_0 be a graph embedded in some surface Σ of Euler genus γ and let G^+ be another graph that might share common vertices with G_0 . We set $G = G_0 \cup G^+$. Let also H be some graph and let $v \in V(H)$. We say that G *contains a graph H as a v -smooth contraction* if $H \leq_c^\phi G$ for some $\phi : V(G) \rightarrow V(H)$ and there exists a closed disk D in Σ such that all the vertices of G that are outside D are exactly the model of v , i.e. $\phi^{-1}(v) = V(G) \setminus (V(G) \cap D)$.

A graph H is a *minor* of a graph G , denoted by $H \leq_m G$, if H is the contraction of some subgraph of G . If we restrict the contraction to edges whose one of the incident vertices has degree exactly two, also called *dissolving* that vertex, then we say that H is

a *topological minor* of G and we denote it by $H \leq_{tm} G$. Moreover, we say that a graph G is a *subdivision* of a graph H , if H can be obtained from G by dissolving vertices.

We say that a graph G is *H -minor-free* when it does not contain H as a minor. We also say that a graph class \mathcal{G} is *H -minor-free* (or, excludes H as a minor) when all its members are H -minor-free.

Graph Minors structure theorem. The proof of our results is using the Excluded Minor Theorem from the Graph Minor theory. Before we state it, we need some definitions.

Definition 1 (*h -nearly embeddable graphs*). Let Σ be a surface and $h > 0$ be an integer. A graph G is *h -nearly embeddable* in Σ if there is a set of vertices $X \subseteq V(G)$ (called *apices*) of size at most h such that graph $G - X$ is the union of subgraphs G_0, \dots, G_h with the following properties

- i) There is a set of cycles C_1, \dots, C_h in Σ such that the cycles C_i are the borders of open pairwise disjoint discs Δ_i in Σ ;
- ii) G_0 has an embedding in Σ in such a way that $G_0 \cap \bigcup_{i=1, \dots, h} \Delta_i = \emptyset$;
- iii) graphs G_1, \dots, G_h (called *vortices*) are pairwise disjoint and for $1 \leq i \leq h$, $V(G_0) \cap V(G_i) \subset C_i$;
- iv) for $1 \leq i \leq h$, let $U_i := \{u_1^i, \dots, u_{m_i}^i\}$ be the vertices of $V(G_0) \cap V(G_i) \subset C_i$ appearing in an order obtained by clockwise traversing of C_i , we call vertices of U_i bases of G_i . Then G_i has a path decomposition $\mathcal{B}_i = (B_j^i)_{1 \leq j \leq m_i}$, of width at most h such that for $1 \leq j \leq m_i$, we have $u_j^i \in B_j^i$.

A *tree decomposition* of a graph G is a pair (\mathcal{X}, T) where T is a tree and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a collection of subsets of $V(G)$ such that:

1. $\bigcup_{i \in V(T)} X_i = V(G)$;
2. for each edge $\{x, y\} \in E(G)$, $\{x, y\} \subseteq X_i$ for some $i \in V(T)$, and
3. for each $x \in V(G)$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T .

The *width* of a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ is $\max_{i \in V(T)} \{|X_i| - 1\}$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G . Furthermore, we call the subsets X_i *bags* of the decomposition and for every $X_i, i \in V(T)$ we denote by \overline{X}_i its closure, i.e. \overline{X}_i is the graph $G[X_i] \cup (\bigcup_{j \in N_T(i)} K[X_i \cap X_j])$.

Observation 2. Let G be a graph and (\mathcal{X}, T) be a tree decomposition of G . Then there exists an $X \in \mathcal{X}$ such that $\text{tw}(\overline{X}) \geq \text{tw}(G)$.

We also need the simple following result.

Lemma 1. *If G is a graph and $X \subseteq V(G)$, then $\text{tw}(G - X) \geq \text{tw}(G) - |X|$.*

We say that a tree decomposition (\mathcal{X}, T) of a graph G is *small* if for every $i, j \in V(T)$, $i \neq j$, $X_i \not\subseteq X_j$.

A simple proof of the following lemma can be found in [3].

Lemma 2. *1. Let G be a graph and (\mathcal{X}, T) be a small tree decomposition of G . Then $|V(T)| \leq |V(G)|$.*

2. Every graph G has a small tree decomposition of width $\text{tw}(G)$.

The following proposition is known as the Graph Minors structure theorem [11].

Proposition 1. *There exists a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for every non-planar graph H with h vertices and every graph G excluding H as a minor there exists a tree decomposition $(\mathcal{G} = \{G_i \mid i \in V(T)\}, T)$ where for every $i \in V(T)$, \overline{G}_i is an $f(h)$ -nearly embeddable graph in a surface Σ on which H cannot be embedded.*

3 Statement of the main result

Let k and r be positive integers where $k, r \geq 2$. The $(k \times r)$ -grid is the Cartesian product of two paths of lengths $k - 1$ and $r - 1$ respectively. A vertex of a $(k \times r)$ -grid is a *corner* if it has degree 2. Thus each $(k \times r)$ -grid has 4 corners. A vertex of a $(k \times r)$ -grid is called *internal* if it has degree 4, otherwise it is called *external*.

We define Γ_k as the following (unique, up to isomorphism) triangulation of the $(k \times k)$ -grid. Let Γ be a plane embedding of the $(k \times k)$ -grid such that all external vertices are on the boundary of the external face. We triangulate internal faces of the $(k \times k)$ -grid such that, in the obtained graph, all the internal vertices have degree 6 and all non-corner external vertices have degree 4. The construction of Γ_k is completed if we connect one corner of degree two with all vertices of the external face (we call this corner *loaded*). We also use notation Γ_k^* for the graph obtained from Γ_k if we remove all edges incident to its loaded vertex that do not exist in its underlying grid.

We define the (k, l) -pyramid to be the graph obtained if we take the disjoint union of a $(k \times k)$ -grid and a clique K_l and then add all edges between the vertices of the clique and the vertices of the grid. We denote the (k, l) -pyramid by $\Pi_{k,l}$.

Walls. A *wall of height k* , $k \geq 1$, is the graph obtained from a $((k+1) \times (2 \cdot k+2))$ -grid with vertices (x, y) , $x \in \{1, \dots, 2 \cdot k+4\}$, $y \in \{1, \dots, k+1\}$, after the removal of the “vertical” edges $\{(x, y), (x, y+1)\}$ for odd $x+y$, and then the removal of all vertices of degree 1. We denote such a wall by W_k .

The *corners* of the wall W_k are the vertices $c_1 = (1, 1)$, $c_2 = (2 \cdot k + 1, 0)$, $c_3 = (2 \cdot k + 1 + (k + 1 \bmod 2), k + 1)$ and $c_4 = (1 + (k + 1 \bmod 2), k + 1)$. We let $C = \{c_1, c_2, c_3, c_4\}$ and we call the pairs $\{c_1, c_3\}$ and $\{c_2, c_4\}$ *anti-diametrical*.

A *subdivided wall* W of height k is a wall obtained from W_k after replacing some of its edges by paths without common internal vertices. We call the resulting graph W a *subdivision* of W_k and the vertices that appear in the wall after the replacement *subdivision* vertices.

The non-subdivision vertices of W are called *original* vertices. The *perimeter* P of a subdivided wall is the cycle defined by its boundary.

The *layers* of a subdivided wall W of height k are recursively defined as follows. The first layer of W is its perimeter. For $i = 2, \dots, \lfloor \frac{k}{2} \rfloor$, the i -th layer of W is the $(i - 1)$ -th layer of the subwall W' obtained from W after removing from W its perimeter and all occurring vertices of degree 1 (see Figure 1).

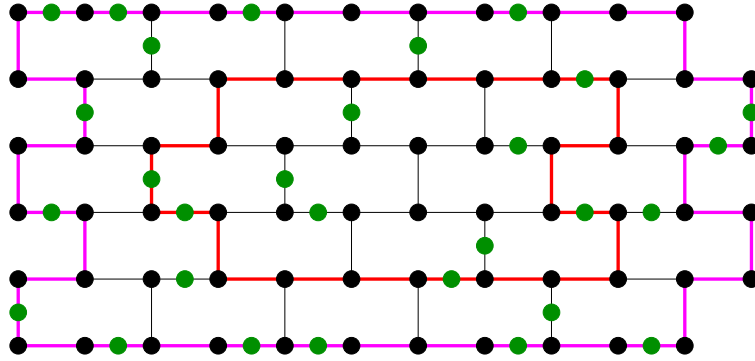


Figure 1: The first (magenta) and second (red) layers of a wall of height 5

If W is a subdivided wall of height k , we call *brick* of W any facial cycle whose non-subdivided counterpart in W_h has length 6. We say that two bricks are *neighbors* if their intersection contains an edge.

Let W_k be a wall. We denote by $P_j^{(h)}$ the shortest path connecting vertices $(1, j)$ and $(2 \cdot k + 2, j)$, and by $P_i^{(v)}$ the shortest path connecting vertices $(i, 1)$ and $(i, k + 1)$ with the assumption that for $i < 2 \cdot k + 2$, $P_i^{(v)}$ contains only vertices (x, y) with $x = i, i + 1$. We call the paths $P_{k+1}^{(h)}$ and $P_1^{(h)}$ the *southern path* of W_k and *northern part* of W_k respectively. Similarly, we call the paths $P_1^{(v)}$ and $P_{2 \cdot k + 1}^{(v)}$ the *western part* of W_k and the *eastern part* of W_k respectively. If W is a subdivision of W_k , we will use the same notation for the paths obtained by the subdivisions of the corresponding paths of W_k .

Compasses and rural deviations. Let W be a subdivided wall in G . Let K' be the connected component of $G \setminus P$ that contains $W \setminus P$. The *compass* K of W in G is the

graph $G[V(K') \cup V(P)]$. Observe that W is a subgraph of K and K is connected. We say that a wall is *flat* in G if its compass K in G has no (c_1, c_3) -path and (c_2, c_4) -path that are vertex-disjoint.

Observation 3. *Let W be a flat wall. Then any subdivision of W is also flat.*

If J is a subgraph of K , we denote by $\partial_K J$ the set of all vertices v such that either $v \in C$ or v is incident with an edge of K that is not in J , i.e.

$$\partial_K J = \{v \in V(J) \mid v \in C \text{ or } \exists e \in E(K) \setminus E(J) : v \in e\}.$$

A *rural division* \mathcal{D} of the compass K is a collection (D_1, D_2, \dots, D_m) of subgraphs of K with the following properties:

1. $\{E(D_1), E(D_2), \dots, E(D_m)\}$ is a partition of non-empty subsets of $E(K)$,
2. for $i, j \in [m]$, if $i \neq j$ then $\partial_K D_i \neq \partial_K D_j$ and $V(D_i) \cap V(D_j) = \partial_K D_i \cap \partial_K D_j$,
3. for each $i \in [m]$ and all $x, y \in \partial_K D_i$ there exists a (x, y) -path in D_i with no internal vertex in $\partial_K D_i$,
4. for each $i \in [m]$, $|\partial_K D_i| \leq 3$, and
5. the hypergraph $H_K = (\bigcup_{i \in [m]} \partial_K D_i, \{\partial_K D_i \mid i \in [m]\})$ can be embedded in a closed disk Δ such that c_1, c_2, c_3 and c_4 appear in this order on the boundary of Δ and for each hyperedge e of H there exist $|e|$ mutually vertex-disjoint paths between e and C in K .

We call the elements of \mathcal{D} *flaps*. A flap $D \in \mathcal{D}$ is *internal* if $V(D) \cap V(P) = \emptyset$.

We are now in position to state the main result of this paper.

Theorem 1. *There exists a computable function f such that for every two graphs H and G and every $k \in \mathbb{N}$, one of the following holds:*

1. H is a minor of G ,
2. $\text{tw}(G) \leq f(h) \cdot k$, where $h = \mathbf{n}(H)$
3. $\exists A \subseteq V(G)$ with $|A| \leq \mathbf{an}(H) - 1$ such that $G \setminus A$ contains as a subgraph a flat subdivided wall W where
 - W has height k and
 - the compass of W has a rural division \mathcal{D} such that each internal flap of \mathcal{D} has treewidth at most $f(h) \cdot k$.

We postpone the proof of Theorem 1 until Section 5.

4 Some auxiliary lemmata

The main results in this section are Lemmata 6 and 10 that will be used for the proof of our main result in Section 5.

4.1 An invariance lemma for flatness

Lemma 3. *Let k be a positive integer and G be a graph that contains $\Gamma_{2 \cdot k+8}$ as a v -smooth contraction. Then G contains as a subgraph a subdivided wall of height k whose compass can be embedded in a closed disk Δ such that the perimeter of W is identical to the boundary of Δ .*

Proof. Assume that $\Gamma_{2 \cdot k+8}$ is a v -smooth contraction of G via ϕ . W.l.o.g. let $V(\Gamma_{2 \cdot k+8}) = \{1, \dots, 2 \cdot k+8\}^2$, where $v = (2 \cdot k+8, 2 \cdot k+8)$. Let R be the set of external vertices of $\Gamma_{2 \cdot k+8}$ and let $G' = G \setminus \bigcup_{x \in R} \phi^{-1}(x)$. As G contains $\Gamma_{2 \cdot k+8}$ as a v -smooth contraction, it follows that G' is embedded inside an open disk Δ' . Moreover G' can be contracted to $\Gamma_{2 \cdot k+6}^*$ via the restriction of ϕ to $V(G')$. From the definition of a wall, it follows that $\Gamma_{2 \cdot k+6}^*$ contains W_{k+2} as a subgraph. As G' contains $\Gamma_{2 \cdot k+6}^*$ as a minor, it follows that G' contains W_{k+2} as a minor. As W_{k+2} has maximum degree 3, it is also a topological minor of G' . Therefore G' contains as a subgraph (embedded in Δ') a subdivided wall of height $k+2$. Among all such subdivided walls, let W_{ex} be the one whose compass has the minimum number of faces inside the annulus $\Phi = \Delta_{\text{ex}} \setminus \Delta \subseteq \Delta'$ where Δ_{ex} and Δ are defined as the closed disks defined so that the boundary of Δ_{ex} is the first layer of W_{ex} and the boundary of Δ is the second one.

Let W be the subdivided wall of G' whose perimeter is the boundary of Δ . By definition, all vertices of the compass K of W are inside Δ . It now remains to prove that the same holds also for the edges of K . Suppose in contrary that $\{x, y\}$ is an edge outside Δ . Clearly, both x and y lie on the boundary of Δ and $\{x, y\}$ is inside the disk Δ^* defined by some brick of W_{ex} that is inside Φ . We distinguish two cases:

Case 1: $\{x, y\}$ are in the same brick, say A of W . Then, there is a path of this brick that can be replaced in W by $\{x, y\}$ and substitute W by a new subdivided wall corresponding to an annulus with less faces, a contradiction.

Case 2: $\{x, y\}$ are not in the same brick of W . Then x and y should belong in neighboring bricks, say B and C respectively. Let A be the unique brick of W_{ex} that contains x and y and w be the unique common vertex in A, B and C . Observe that there a path P_B of B connecting x and w and a path P_C of C connecting y and w . Then we substitute W by a new wall as follows: we replace w by x , P_C by $\{x, y\}$, and see P_B as a subpath of the common path between B and C . Again, the new wall corresponds to an annulus with less faces, a contradiction. \square

Lemma 4 ([4]). *There is a function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that if G is a graph h -nearly embedded in a surface of Euler genus γ without apices, where $\mathbf{tw}(G) \geq f(\gamma, h) \cdot k$, then G contains as a v -smooth contraction the graph Γ_k with the loaded corner v .*

Lemma 5. *Let h be a positive integer and G be a graph that contains a flat subdivided wall W of height h . If K_3 is a subgraph of the compass of W then after applying a ΔY -transformation in K_3 the resulting graph also contains a flat subdivided wall W' of height h as a subgraph. Moreover, W' is isomorphic to a subdivision of W .*

Proof. We examine the non-trivial case where $E(K_3) \cap E(W) \neq \emptyset$. Observe that, as W does not contain triangles, $|E(K_3) \cap E(W)| < 3$. In what follows we denote by x, y, z the vertices of K_3 , w the vertex that appears after the transformation, and distinguish the following cases.

Case 1. K_3 and W have exactly one common edge, say $\{x, y\}$. As w is a new vertex, the path (x, w, y) that appears after the ΔY -transformation has no common internal vertices with W . In this case, we replace the edge $\{x, y\}$ in W by the edges $\{x, w\}, \{w, y\}$.

Case 2. K_3 and W have exactly two common edges, say $\{x, y\}$ and $\{x, z\}$. We distinguish the following two subcases.

Subcase 2.1. x is an original vertex and x is not a corner of W . Let q be the third vertex in the neighborhood of x . Observe that the ΔY -transformation is equivalent to removing the edge $\{y, z\}$, which is not an edge of the wall, and subdividing the edge $\{x, q\}$. Then the lemma follows from Observation 3.

Subcase 2.2. x is not an original vertex or x is a corner. Then the ΔY -transformation is equivalent to removing the edge $\{y, z\}$, which is not an edge of W , and then substituting $\{y, x\}$ by $\{y, w\}$ and $\{x, z\}$ by $\{w, z\}$.

Observe that in all the above cases the resulting wall W' remains flat and is isomorphic to a subdivision of W and the lemma follows. \square

By applying inductively Observation 3 and Lemma 5 we derive the following.

Lemma 6. *Let h be a positive integer and G be a graph that contains a flat subdivided wall W of height h as a subgraph. If we apply a sequence of subdivisions or ΔY -transformations in G , then the resulting graph will contain a flat subdivided wall W' of height h as a subgraph. Moreover, W' is isomorphic to a subdivision of W .*

4.2 Pyramids and treewidth

Combining Proposition (1.5) in [12] with Euler's formula for planar graphs we obtain the following.

Lemma 7. *If G is a planar graph then G is isomorphic to a minor of the $(14 \cdot \mathbf{n}(G) - 24) \times (14 \cdot \mathbf{n}(G) - 24)$ -grid.*

From the above lemma we obtain the following.

Lemma 8. *Let h be an integer. If G is an h -apex graph then G is isomorphic to a minor of $\Pi_{14 \cdot (\mathbf{n}(G) - h) - 24, h}$.*

Lemma 9. *Let G be the graph obtained by a $((k + \lceil \sqrt{h} \rceil) \times (k + \lceil \sqrt{h} \rceil))$ -grid if we make its vertices adjacent to a set A of h new vertices. Then G contains $\Pi_{k, h}$ as a minor.*

Proof. We denote by G' the grid used for constructing G and let G_1 and G_2 two disjoint subgraphs of G' where G_1 is isomorphic to a $(k \times k)$ -grid and G_2 is isomorphic to a $(\alpha \times \alpha)$ -grid where $\alpha = \lceil \sqrt{h} \rceil$. Remove from G all vertices not in $A \cup V(G_1) \cup V(G_2)$. Then remove all edges of G' incident to $V(G_2)$ and notice that in the remaining graph F , the vertices in $A \cup V(G_2)$ induce a graph isomorphic to K_{h, α^2} which, in turn, can be contracted to a clique on the vertices of A . Applying the same contractions in F one may obtain $\Pi_{k, h}$ as a minor of G . \square

Lemma 10. *Let G, H be graphs such that H is not a minor of G and there exists a set $A \subseteq V(G)$ such that $G \setminus A$ contains a wall W of height $g(h) \cdot (k + 1) - 1$ as a subgraph, where $g(h) = 14 \cdot (h - \mathbf{an}(h)) + \lceil \sqrt{\mathbf{an}(h)} \rceil - 24$ and $h = \mathbf{n}(H)$. If $|A| \geq \mathbf{an}(h)$ then there exists an $A' \subseteq A$ such that $|A'| < |A|$ and $G \setminus A'$ contains a wall $W' \subseteq W$ of height k as a subgraph. Moreover, if K' is the compass of W' in $G \setminus A'$ then $V(K') \cap A = \emptyset$.*

Proof. Let $A = \{\alpha_i \mid i \in [|A|]\}$ and $P_{g(h)} = \{W_{(m, l)} \mid (m, l) \in [g(h)]^2\}$ be a collection of $(g(h))^2$ disjoint subwalls $W_{(m, l)}, (m, l) \in [g(h)]^2$ of W with height k . For every $(m, l) \in [g(h)]^2$, we denote by $K_{(m, l)}$ the compass of $W_{(m, l)}$ in $G \setminus A$ and let $q_{(m, l)} = (q_{(m, l)}^1, q_{(m, l)}^2, \dots, q_{(m, l)}^{|A|})$ be the binary vector where for every $j \in |A|$,

$$q_{(m, l)}^j = \begin{cases} 1 & \text{if } \exists v \in V(K_{(m, l)}) : \{v, \alpha_j\} \in E(G) \\ 0 & \text{if } \forall v \in V(K_{(m, l)}) : \{v, \alpha_j\} \notin E(G) \end{cases}$$

We claim that there exists an $(m', l') \in [g(h)]^2$ such that $q_{(m', l')} \neq (1, 1, \dots, 1)$. Indeed, assume in contrary, that for every $(m, l) \in [g(h)]^2$, $q_{(m, l)} = (1, 1, \dots, 1)$. We will arrive to a contradiction by showing that H is a minor of G . For this, consider the graph

$$G' = G[V(W) \cup \bigcup_{(m, l) \in [g(h)]^2} V(K_{(m, l)})] \subseteq G.$$

For every $(m, l) \in [g(h)]^2$, contract each $K_{(m, l)}$ to a single vertex and this implies the existence of a $(g(h) \times g(h))$ -grid as a minor of G' and therefore of $G \setminus A$ as well. Moreover, for each vertex v of this grid it holds that each vertex in A is adjacent to some vertex of the model of v , therefore G contains the graph J obtained after we take a $(g(h) \times g(h))$ -grid and connect all its vertices with $\mathbf{an}(h)$ new vertices. From Lemma 9, G contains $\Pi_{14 \cdot (\mathbf{n}(h) - \mathbf{an}(h)) - 24, \mathbf{an}(h)}$ as minor. Applying now Lemma 8, we obtain that G contains H as a minor, a contradiction. Therefore, there exist $(m', l') \in [g(h)]^2$ and $j_0 \in [|A|]$ such that $q_{(m', l')}^{j_0} = 0$. The lemma follows for $A' = A \setminus \{\alpha_{j_0}\}$ and $W' = W_{(m', l')}$. \square

5 The main proof

Given a tree decomposition $\mathcal{T} = (\mathcal{X} = \{X_i \mid i \in V(T)\}, T)$ of a graph G a vertex $i_0 \in V(T)$ and a set of vertices $I \subseteq N_T(i_0)$, we define $\mathcal{T}_{i_0, I}$ as the collection of connected components of $T \setminus i_0$ that contain vertices of I . Given a subtree Y of T , we define $G_Y = G[\cup_{i \in V(Y)} X_i]$ and $\overline{G}_Y = \cup_{i \in V(Y)} \overline{X}_i$.

Observation 4. *Given a tree decomposition $\mathcal{T} = (\mathcal{X} = \{X_i \mid i \in V(T)\}, T)$ of a graph G , a vertex $i_0 \in V(T)$, and a set of vertices $I \subseteq N_T(i_0)$, it holds that for every $T_1, T_2 \in \mathcal{T}_{i_0, I}$, $\overline{G}_{T_1} \cap \overline{G}_{T_2}$ is a complete graph.*

Given a family of graphs \mathcal{F} , a graph G and a set of vertices $S \subseteq V(G)$, we define the class $\mathcal{F}_{S, G}^*$ as the collection of the connected components in the graphs of $\mathcal{F} \setminus S$ and the class $\mathcal{F}_{S, G}$ as the set of graphs in $\mathcal{F}_{S, G}^*$ that have some common vertex with $G \setminus S$. We say that two graphs $G_1, G_2 \in \mathcal{F}_{S, G}$ are G -equivalent if $V(G_1) \cap V(G \setminus S) = V(G_2) \cap V(G \setminus S)$ and let $\mathcal{F}_{S, G}^1, \dots, \mathcal{F}_{S, G}^\rho$ be the equivalence classes defined that way. We denote by $\mathcal{P}_{\mathcal{F}, S, G} = \{\mathbf{U}\mathcal{F}_{S, G}^1, \dots, \mathbf{U}\mathcal{F}_{S, G}^\rho\}$, i.e. for each equivalence class $\mathcal{F}_{S, G}^i$ we construct a graph in $\mathcal{P}_{\mathcal{F}, S, G}$, by taking the union of the graphs in $\mathcal{F}_{S, G}^i$.

Proof of Theorem 1. Let G be a graph that excludes H as a minor. By Proposition 1, there is a computable function f_1 such that there exists a tree decomposition

$$\mathcal{T} = (\mathcal{X} = \{X_i \mid i \in V(T)\}, T)$$

of G , where for every $i \in V(T)$, the graphs \overline{X}_i are $f_1(h)$ -nearly-embeddable in a surface Σ of genus $f_1(h)$. Among all such tree-decompositions we choose $\mathcal{T} = (\mathcal{X}, T)$ such that

- (i) \mathcal{T} is small.
- (ii) subject to (i) \mathcal{T} has maximum number of nodes.
- (iii) subject to (ii) the quantity $\sum_{\substack{i, j \in V(T) \\ i \neq j}} |X_i \cap X_j|$ is minimized.

Notice that, from Lemma 2, the condition (i) guaranties the possibility of the choice of condition (ii). We use the notation \overline{G} to denote the graph \overline{G}_T and we call the edges of $E(\overline{G}) \setminus E(G)$ *virtual*.

Let $w : V(T) \rightarrow \mathbb{N}$ such that $w(i) = \mathbf{tw}(\overline{X}_i)$. Observation 1 and Observation 2 imply that there exists a vertex $i_0 \in V(T)$ such that $\mathbf{tw}(\overline{X}_{i_0}) \geq \mathbf{tw}(G)$ and at most one of the connected components of $T \setminus i_0$ contains a vertex j such that $w(j) > w(i_0)$. We denote by A_{i_0} the set of apices of the graph \overline{X}_{i_0} and by F the graph $\overline{X}_{i_0} \setminus A_{i_0}$ (notice that $F \subseteq \overline{G}$ but F is not necessarily a subgraph of G as F may contain virtual edges).

Claim 1. For every connected component Y of $T \setminus \{i_0\}$, there is a vertex in $G_Y \setminus X_{i_0}$ connected with $X_{i_0} \cap V(G_Y)$ with $|X_{i_0} \cap V(G_Y)|$ vertex disjoint paths whose internal vertices belong in $G_Y \setminus X_{i_0}$.

Proof of Claim 1. First, observe that, from (ii) the graph $G_Y \setminus X_{i_0}$ is connected. Then, from (iii), it follows that there is a vertex in $G_Y \setminus X_{i_0}$ connected with $X_{i_0} \cap V(G_Y)$ with $|X_{i_0} \cap V(G_Y)|$ vertex disjoint paths.

From Lemma 1 and the choice of i_0 holds that

$$\mathbf{tw}(F) = \mathbf{tw}(\overline{X}_{i_0} \setminus A_{i_0}) \geq \mathbf{tw}(\overline{X}_{i_0}) - |A_{i_0}| \geq \mathbf{tw}(G) - |A_{i_0}|. \quad (1)$$

Recall that $|A_{i_0}| \leq f_1(h)$. Let f_2 be the two-variable function of Lemma 4. We define the two-variable function f_3 and the one-variable functions f_4 and f_5 such that

$$\begin{aligned} f_5(h) &= 14 \cdot (h - \mathbf{an}(h)) + \lceil \sqrt{\mathbf{an}(h)} \rceil - 24 \\ f_4(h) &= f_5(h)^{|A_{i_0}| - \mathbf{an}(h) + 1} \\ f_3(h, k) &= f_2(f_1(h), f_1(h)) \cdot (4k \cdot f_4(h) + 12) + f_1(h) \end{aligned}$$

As F is $f_1(h)$ -nearly embeddable in Σ and does not contain any apices, from (1) and Lemma 4, we obtain that if $\mathbf{tw}(G) \geq f_3(h, k)$ then F contains the graph $Q = \Gamma_{4k \cdot f_4(h) + 12}$ as a v -smooth contraction. By Lemma 3, it follows that F contains as a subgraph a flat subdivided wall W' of height $2k \cdot f_4(h) + 2$ whose compass K' in F can be embedded in a closed disk Δ such that the perimeter of W' is identical to the boundary of Δ . Let

$$I' = \{i \in N_T(i_0) \mid X_i \cap V(K') \neq \emptyset\}.$$

In other words I' corresponds to all nodes of the tree decomposition \mathcal{T} that have vertices “inside” the compass of the subdivided wall W' in F . Clearly, every clique in K' has size at most 4. Furthermore, K' does not contain any K_4 as a subgraph. Indeed, if so, one of its triangles would be a separator of G , contradicting minimality of condition (iii) in the definition of the decomposition \mathcal{T} . To see this, just replace X_{i_0} in \mathcal{T} by $X_{i_0} \setminus \{z\}$ where z is the vertex of the clique that is not in the separating triangle. Recall, now, that for every $i \in I'$, $F[V(K') \cap X_i] \subseteq K'$ is a clique. Thus,

$$\text{for every } i \in I', |V(K') \cap X_i| \leq 3. \quad (2)$$

Recall, also, that there exists at most one tree in $\mathcal{T}_{i_0, I'}$, say T' , that contains a vertex i_1 with $w(i_1) > w(i_0)$. Let $\mathcal{W}' = \{W'_1, W'_2, W'_3, W'_4\}$ be the collection of vertex disjoint subwalls of W' of height $f_4(h) \cdot k$ not meeting the vertices of $P_{k \cdot f_4(h) + 2}^{(h)}$ and $P_{2k \cdot f_4(h) + 3}^{(v)}$ (see Figure 5).

From (2), X_{i_1} has at most 3 vertices in common with K' , therefore there exists a subwall $\tilde{W} \in \mathcal{W}'$ of height $f_4(h) \cdot k$, with compass \tilde{K} in F such that $V(\tilde{K}) \cap V(G_{T'}) = \emptyset$. Consequently, if we set

$$\tilde{I} = \{i \in N_T(i_0) \mid X_i \cap V(\tilde{K}) \neq \emptyset\}$$

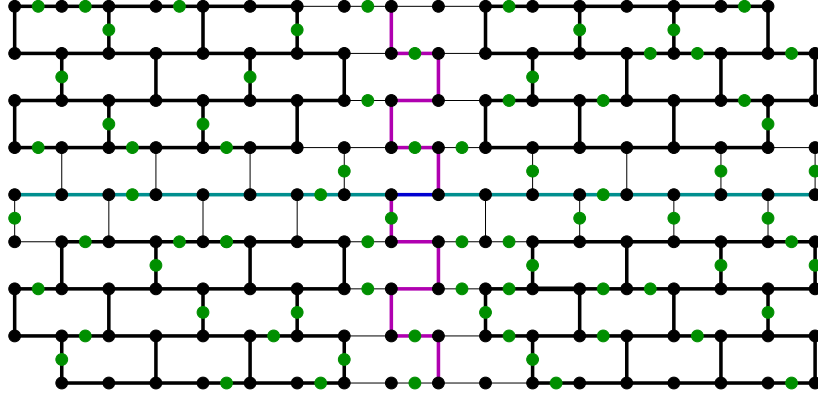


Figure 2: The paths $P_{k \cdot f_4(h)+2}^{(h)}$ (cyan) and $P_{2k \cdot f_4(h)+3}^{(v)}$ (magenta) and the corresponding walls for $k = 1$ and $f_4(h) = 3$.

we have that $\tilde{I} \subseteq I' \setminus \{i_1\}$ and for every tree $\tilde{T} \in \mathcal{T}_{i_0, \tilde{I}} \subseteq \mathcal{T}_{i_0, I'} \setminus \{T'\}$ it holds that $\max\{w(i) \mid i \in V(\tilde{T})\} \leq f_3(h, k)$. Therefore, for every $\tilde{T} \in \mathcal{T}_{i_0, \tilde{I}}$, $\mathbf{tw}(\overline{G_{\tilde{T}}}) \leq f_3(h, k)$. As $G_{\tilde{T}}$ is a subgraph of $\overline{G_{\tilde{T}}}$, it follows that

$$\text{for every } \tilde{T} \in \mathcal{T}_{i_0, \tilde{I}}, \quad \mathbf{tw}(G_{\tilde{T}}) \leq f_3(h, k). \quad (3)$$

From (2), it follows that for every $\tilde{T} \in \mathcal{T}_{i_0, \tilde{I}}$, the vertices in $V(G_{\tilde{T}}) \cap V(\tilde{K})$ induce a clique in \tilde{K} with at most 3 vertices where some of its edges may be virtual.

Let $\tilde{V} = V(F) \setminus V(\tilde{K})$ and $\mathcal{F}' = \{G_{\tilde{T}} \mid \tilde{T} \in \mathcal{T}_{i_0, \tilde{I}}\}$. Notice that $\tilde{K} = F \setminus \tilde{V}$. We denote by \mathcal{F} the class $\mathcal{P}_{\mathcal{F}', \tilde{V}, F}$.

We call the edges in $\tilde{E} = E(\tilde{K}) \setminus E(G)$ *useless*. We also call all vertices in $V(\mathbf{UF}) \setminus V(\tilde{K})$ *flying* vertices. The non-flying vertices of a graph R in \mathcal{F} are the *base* of R . Notice that, by the definition of \mathcal{F} , each graph R in \mathcal{F} is a subgraph of the union of some graphs of \mathcal{F}' . From Observation 4 and (3), It follows that

- (a) all graphs in \mathcal{F} have treewidth at most $f_3(h, k)$

Observation 4 and (2) yields that

- (b) the base vertices of each R induce a clique of size 1, 2, or 3 in \tilde{K} .

Also, from Claim 1 and the fact that $\tilde{V} \cup A_{i_0} \subseteq X_{i_0}$, we have that

- (c) each pair of vertices of some graph in \mathcal{F} are connected in G by a path whose internal vertices are flying.

Note that each clique mentioned in (b) may contain useless edges. Moreover, from (c), all virtual edges of \tilde{K} are edges of such a clique. Let $\tilde{G} = (V(G), \tilde{E} \cup E(G))$, i.e. we add in G all useless edges.

It now follows that $\tilde{G} \setminus A_{i_0}$ contains the wall \tilde{W} as a subgraph and the compass of \tilde{W} in $\tilde{G} \setminus A_{i_0}$ is

$$\tilde{K}^+ = \tilde{K} \cup \bigcup_{R \in \mathcal{F}} R$$

Notice that the wall \tilde{W} remains flat in \tilde{G} . Indeed, suppose that Q_1 and Q_2 are two vertex disjoint paths between the two anti-diametrical corners of \tilde{W} such that the sum of their lengths is minimal. As not both of Q_1 and Q_2 may exist in \tilde{K} , some of them, say Q_1 contains some flying vertex. Let R be the graph in \mathcal{F} containing that vertex. Then there are two vertices x and y of the base of R met by Q_1 . From (b), $\{x, y\}$ is an edge of \tilde{K}^+ and we can substitute the portion of Q_1 that contains flying vertices by $\{x, y\}$, a contradiction to the minimality of the choice of Q_1 and Q_2 .

Let $\tilde{E}^+ = E(\tilde{K}^+) \setminus E(\mathbf{UF})$, i.e. \tilde{E}^+ is the set of edges of \tilde{K} not contained in any graph R of \mathcal{F} . It follows that all useless edges are contained in \tilde{E}^+ , i.e.

$$\tilde{E} \subseteq \tilde{E}^+ \quad (4)$$

For every $e \in \tilde{E}^+$, we denote by \tilde{G}_e the graph formed by the edge e (i.e. the graph $\tilde{G}[e]$) and let $\mathcal{E} = \{\tilde{G}_e \mid e \in \tilde{E}^+\}$. We set $\tilde{\mathcal{D}}^+ = \mathcal{F} \cup \mathcal{E}$. Notice that,

$$\text{For every graph } R \in \mathcal{F}, \quad \partial_{\tilde{K}^+} R \text{ is the base of } R \quad (5)$$

$$\text{For every graph } \tilde{G}_e \in \mathcal{E}, \quad \partial_{\tilde{K}^+} \tilde{G}_e = V(\tilde{G}_e) \quad (6)$$

Claim 2. $\tilde{\mathcal{D}}^+ = \mathcal{F} \cup \mathcal{E}$ is a rural division of \tilde{K}^+ .

Proof of Claim 2. Properties 1 and 2, follow from the construction of the graphs in \mathcal{F} and \mathcal{E} . Moreover, Properties 3 and 4 follow from (c) and (b) respectively. For Property 5, recall that \tilde{W} is a subwall of W' whose compass K' in F can be embedded in a closed disk Δ such that the perimeter of W' is identical to its boundary. This implies that \tilde{K} can be embedded in a closed disk $\tilde{\Delta} \subseteq \Delta$ such that the corners c_1, c_2, c_3 , and c_4 of \tilde{W} appear in this order on its boundary. We now consider the following hypergraph:

$$\tilde{H}^+ = (\mathbf{U}\{\partial_{\tilde{K}^+} D \mid D \in \tilde{\mathcal{D}}^+\}, \{\partial_{\tilde{K}^+} D \mid D \in \tilde{\mathcal{D}}^+\}).$$

Notice that $V(\tilde{H}^+) = V(\tilde{K})$. We can now construct $I(\tilde{H}^+)$ by applying, for each $D \in \tilde{\mathcal{D}}^+$, the following transformations on the planar graph \tilde{K} .

- If $|\partial_{\tilde{K}^+} D| = 1$, we add a new vertex and an edge that connects it the unique vertex of $\partial_{\tilde{K}^+} D$.

- If $|\partial_{\tilde{K}^+} D| = 2$, we subdivide the edge of $\tilde{K}[\partial_{\tilde{K}^+} D]$ (recall that $\tilde{K}[\partial_{\tilde{K}^+} D]$ is isomorphic to K_2).
- If $|\partial_{\tilde{K}^+} D| = 3$, we apply a ΔY -transformation in $\tilde{K}[\partial_{\tilde{K}^+} D]$ (recall that $\tilde{K}[\partial_{\tilde{K}^+} D]$ is isomorphic to K_3).

From Observation 3 and Lemma 5 follows that the obtained graph remains embedded in $\tilde{\Delta}$. It now remains to show that for each $e \in E(\tilde{H}^+)$ there exist $|e|$ vertex-disjoint paths between $|e|$ and C in \tilde{K}^+ . Notice that for each $e \in E(H^+)$ the vertices of e belong to \tilde{K} . Finally, there are $|e|$ paths between e and C , otherwise we would have a contradiction to the choice of the tree-decomposition (assumption (iii)). Therefore all conditions required for Claim 2 hold.

Our aim now is to find in $G \setminus A_{i_0}$ a flat subdivided wall \widehat{W} of height $f_4(h) \cdot k$. From (b),(c), and (5), all the useless edges of \tilde{K} are induced by the sets $\partial_{\tilde{K}^+} R$, $R \in \mathcal{F}$ where $\tilde{K}[\partial_{\tilde{K}^+} R]$ is isomorphic to either K_2 or K_3 . Our next step is to prove that, in both cases, we may find a flat subdivided wall in $G \setminus A_{i_0}$ of height $f_4(h) \cdot k$ that does not contain any useless edges.

Case 1. $\tilde{K}[\partial_{\tilde{K}^+} R]$ is isomorphic to K_2 . Then, from (c), there exists a path in R whose endpoints are the vertices of $\partial_{\tilde{K}^+} R$ and such that its internal vertices are flying.

Case 2. $\tilde{K}[\partial_{\tilde{K}^+} R]$ is isomorphic to K_3 . Claim 1, combined with the facts that $\tilde{V} \cup A_{i_0} \subseteq X_{i_0}$ and that $\forall R \in \mathcal{F} \ \partial_{\tilde{K}^+} R \subseteq X_{i_0}$, imply that there exists a flying vertex v_R in R and vertex-disjoint paths between v_R and the vertices of $\partial_{\tilde{K}^+} R$ whose internal vertices are also flying.

The above case analysis implies that for each $R \in \mathcal{F}$ the edge $\{x, y\}$ or the triangle with vertices $\{x, y, z\}$, induced by $\partial_{\tilde{K}^+} R$ may be substituted, using subdivisions or ΔY -transformations by a flying path between x and y or by three flying paths from a flying vertex v_R to x, y , and z respectively. As all edges of these paths are flying, they cannot be useless and therefore they exist also in $G \setminus A_{i_0}$. We are now in position to apply Observation 3 and Lemma 6 and obtain that $\tilde{G} \setminus A_{i_0}$ contains a flat subdivided wall \widehat{W} of height $f_4(h) \cdot k$ such that

(I) $E(\widehat{W}) \cap \tilde{E} = \emptyset$ (recall that \tilde{E} is the set of the useless edges) and

(II) \widehat{W} is isomorphic to a subdivision of \tilde{W} .

Therefore, from (I), \widehat{W} is a flat subdivided wall of height $f_4(h) \cdot k$ in $G \setminus A_{i_0}$.

Let \tilde{C} and \widehat{C} be the corners of \tilde{W} and \widehat{W} respectively. We denote by σ be the bijection from \tilde{C} to \widehat{C} induced by the isomorphism in (II). We also enhance σ by defining $\phi = \sigma \cup \{(x, x) \mid x \in V(\tilde{W}) \setminus C(\tilde{W})\}$.

Let \widehat{K} be the compass of \widehat{W} in $G \setminus A_{i_0}$. We claim that

$$\widehat{\mathcal{D}} = \{D \cap \widehat{K} \mid D \in \widetilde{\mathcal{D}}^+\}$$

is a rural division of \widehat{K} . This is easy to verify in what concerns conditions (1–4). Condition (5) follows by the observation that the mapping ϕ , defined above, is an isomorphism between $H_{\widehat{K}^+}$ and $H_{\widehat{K}}$.

So far, we have found a flat subdivided wall \widehat{W} in $G \setminus A_{i_0}$ and a rural division of its compass \widehat{K} . As each flap in $\widehat{\mathcal{D}}$ is a subgraph of a flap in $\widetilde{\mathcal{D}}^+$ we obtain that all flaps in $\widehat{\mathcal{D}}$ have treewidth at most $f_3(h, k)$. By applying Lemma 10 $|A_{i_0}| - \mathbf{an}(h) + 1$ times, it follows that there exists a set $A \subseteq A_{i_0}$, such that $|A| \leq \mathbf{an}(h) - 1$ and $G \setminus A$ contains a flat subdivided wall W of height k such that $W \subseteq \widehat{W}$. Moreover, $V(K) \cap A_{i_0} = \emptyset$, where K is the compass of W in $G \setminus A$. As above,

$$\mathcal{D} = \{D \cap K \mid D \in \widehat{\mathcal{D}}\}$$

is a rural division of K where all of its flaps have treewidth at most $f_3(h, k)$. The theorem follows as f_3 is a linear function of k . \square

The following corollary gives a more precise description of the structure of apex minor free graphs.

Corollary 1. *There exists a computable function f such that for every two graphs H and G , where H is an apex graph and every $k \in \mathbb{N}$, one of the following holds:*

1. $\mathbf{tw}(G) \leq f(h) \cdot k$, where $h = |V(H)|$
2. H is a minor of G ,
3. G contains a flat subdivided wall W where
 - W has height k and
 - the compass of W has a rural division \mathcal{D} such that each internal flap of \mathcal{D} has treewidth at most $f(h) \cdot k$.

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